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Exactly solvable three-level quantum dissipative systems via bosonization of fermion gas-impurity models

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Abstract

We study the relationship between one-dimensional fermion gas-impurity models and quantum dissipative systems, via the method of constructive bosonization and unitary transformation. Starting from an anisotropic Coqblin–Schrieffer model, a new, exactly solvable, three-level quantum dissipative system is derived as a generalization of the standard spin- $\frac{1}{2}$ spin-boson model. The new system has two environmental oscillator baths with ohmic coupling, and admits arbitrary detuning between the three levels. All tunnelling matrix elements are equal, up to one complex phase which is itself a function of the longitudinal and transverse couplings in the integrable limit. Our work underlines the importance of re-examining the detailed structure of fermion-gas impurity models and spin chains, in the light of connections to models for quantum dissipative systems.

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1. Introduction

The understanding of how quantum systems behave in complex environments continues to be an important fundamental problem, with applications ranging from quantum computation and quantum information to nanodevices and biological systems at the molecular level. Many insights into the system dynamics, thermodynamics, critical behaviour and entanglement properties can be gained by investigating simplified models of such quantum dissipative

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systems (QDS) [1–4] with analytical techniques. A poignant example is provided by the well-known equivalence between the spin- $\frac{1}{2}$ anisotropic Kondo model (AKM) [2, 5] and a particular case of the so-called spin-boson model [6]—or two-level quantum dissipative system [2, 6]. In this paper we propose a new model, providing an extension of this correspondence to a three-level QDS, also derivable via the method of constructive bosonization [7, 8] and mapping from an exactly solvable, one-dimensional fermion gas-impurity system. The extension is achieved by bosonization of a fermion gas model with three-component fermions, starting with an *anisotropic* form of a Coqblin–Schrieffer model [9]. We demonstrate the exact solvability of this model, in the context of the standard analysis via coordinate wavefunctions, by verifying that the scattering data can be parametrized by an R -matrix of standard trigonometric type. As a three-level QDS, the model admits arbitrary detuning between the three energy levels, but the bath couplings take on special values in the integrable limit. Moreover, all tunnelling matrix elements are equal, up to one complex phase which is itself a function of the longitudinal and transverse couplings in the integrable limit. In section 2 we propose the new Hamiltonian, and outline the method of bosonization and mapping in subsections 2.1 and 2.2, leading to a new three-level dissipative system model whose features we discuss. In section 2.3 we show that the model is exactly solvable, and the final section 3 includes some comments on potential applications, and indicates directions for further work as concluding remarks. Remarks on notational details and some technical derivations are given in the appendix.

2. Three-level quantum dissipative system

We extend the well-known [2, 6] equivalence between the standard (XXZ -type), spin- $\frac{1}{2}$ AKM model, and a particular case of the spin- $\frac{1}{2}$ spin-boson model—the simplest possible two-level QDS [6]. In order to motivate and present the extension, we first briefly introduce these two models and their parameters. In second-quantized form, the AKM Hamiltonian is

$$\begin{aligned}
H_{AKM} &= \sum_p \hbar v_F p (:c_{p\uparrow}^\dagger c_{p\uparrow} + c_{p\downarrow}^\dagger c_{p\downarrow}:) + J_{\parallel} \sum_{p,p'} (:c_{p\uparrow}^\dagger c_{p'\uparrow} - c_{p\downarrow}^\dagger c_{p'\downarrow}:) S_z \\
&\quad + \frac{1}{2} J_{\perp} \sum_{p,p'} (c_{p\downarrow}^\dagger c_{p'\uparrow} S_+ + c_{p\uparrow}^\dagger c_{p'\downarrow} S_-) \\
&\equiv \sum_{p,\alpha} \hbar v_F p :c_{p\alpha}^\dagger c_{p\alpha}: + J_{\parallel} \sum_{p,\alpha,p',\alpha'} :c_{p\alpha}^\dagger (\sigma_z)_{\alpha\alpha'} c_{p'\alpha'}: S_z \\
&\quad + \frac{1}{2} J_{\perp} \sum_{p,\alpha,p',\alpha'} (c_{p\alpha}^\dagger (\sigma_-)_{\alpha\alpha'} c_{p'\alpha'} S_+ + c_{p\alpha}^\dagger (\sigma_+)_{\alpha\alpha'} c_{p'\alpha'} S_-). \tag{1}
\end{aligned}$$

Here, $c_{p\alpha}^\dagger$, $c_{p\alpha}$ are respectively the creation and annihilation operators for spin- $\frac{1}{2}$ electrons with wavenumber p , and energy $\hbar v_F p$ linearized about the Fermi level ($p = 2\pi n_p/L$, $n_p \in \mathbb{Z}$, for system length L and periodic boundary conditions), while $\alpha, \beta = 1, 2$ or \uparrow, \downarrow are the spin orientations. The second form of (1) is given to facilitate comparisons with the three-level extension of the model, where Gell–Mann matrices will take the role of the Pauli matrices. The symbol $:\dots:$ stands for operator normal ordering relative to the Fermi sea, the vacuum state $|0\rangle$ annihilated by both c_p ($p > 0$) and c_p^\dagger ($p \leq 0$). The electrons interact with a localized magnetic impurity atom (taken to be at the origin in coordinate space) via (anisotropic) coupling between the spin density at the origin with the impurity spin. For clarity its spin operators are written as $S_z, S_{\pm} \equiv S_x \pm iS_y$ rather than as Pauli matrices $\frac{1}{2}\sigma$. Finally, to (1) can be added an external magnetic interaction of the form hS_z , $h := \mu\mu_B B$ for impurity magnetic moment μ .

The spin- $\frac{1}{2}$ spin-boson, or two-level QDS, Hamiltonian is [6]

$$H_{SB} = \frac{1}{2}\varepsilon\sigma_z - \frac{1}{2}\hbar\Delta\sigma_x + \sum_i \hbar\omega_i b_i^\dagger b_i + \sigma_z \sum_i \hbar C_i (b_i + b_i^\dagger). \quad (2)$$

In (2), the two-level system has detuning (level- or well-asymmetry) parameter ε , and the tunnelling amplitude Δ between the two levels or wells. The index i labels a set of harmonic oscillators playing the role of the environment or ‘bath’, with energies determined by the associated frequencies ω_i (the zero-point contributions $\frac{1}{2}\hbar\omega_i$ have been removed by an appropriate shift). The bath interactions with the two-level system, with strengths given by the coupling constants C_i , affect the energy in the upper and lower levels (which are eigenstates of σ_z). The overall influence of the oscillators on the reduced system is described by the spectral density $J(\omega) := \sum_i C_i^2 \delta(\omega - \omega_i)$ [10]. For the cases of interest, it has the so-called *ohmic* form, $J(\omega) \propto \omega e^{-(\omega/\omega_c)}$, where ω_c is the cutoff frequency. An important implicit parameter which is critical to the behaviour of the QDS is the ohmic coupling α , the proportionality constant which determines the strength of this relationship between $J(\omega)$ and ω (which is of course linear, for $\omega \ll \omega_c$).

It is well known that the AKM model (1), which is also closely related to the XXZ Heisenberg spin chain, is exactly solvable [11, 12], and so this in-principle complete analytical access to all details of the spectrum, eigenstates and correlation functions is conferred, through the transcription via bosonization and an associated unitary transformation, on the spin-boson model itself. In this paper we expand on the dictionary of such exactly solvable fermion gas systems, which may be brought into QDS form, presenting a new, *three-level* QDS. Starting with an appropriate 1D fermion gas model, we reiterate the steps of the standard recipe for the bosonization mapping and unitary transformation which established the equivalence between (1) and (2). Finally, we verify using the standard coordinate space approach that the starting model is indeed in the exactly solvable class, thus underlining the utility of the new three level QDS model.

The starting point is an equivalent of the spin- $\frac{1}{2}$ AKM, but for *three*-component fermions rather than spin- $\frac{1}{2}$. In magnetic systems, multicomponent fermions find applications in the Coqblin–Schrieffer (C–S) model [9]; for the transcription to a three-level QDS, we shall require in the fermionic picture, an extended parametrization of the interaction between the local and impurity ‘spins’ with additional terms of ‘anisotropic’ type: an ‘AC–S’ model. The full Hamiltonian thus contains the free-fermion kinetic term, external magnetic field interactions and the analogue of both transverse and longitudinal interaction terms between the localized fermion ‘spin’ operators at the origin, and those of the impurity atom:

$$H_{ACS}^F = \sum_{p,\alpha=1}^3 \hbar v_F p : c_{p\alpha}^\dagger c_{p\alpha} : + \sum_{\alpha} h_{\alpha} S_{\alpha\alpha} + \sum_{p,p',\alpha} J_{\parallel} : c_{p\alpha}^\dagger c_{p'\alpha} : S_{\alpha\alpha} + J_{\perp} \sum_{p,p',\alpha<\beta} (e^{i\zeta_{\alpha\beta}} c_{p\beta}^\dagger c_{p'\alpha} S_{\alpha\beta} + \text{h.c.}). \quad (3)$$

Here p is the fermion wavenumber as before; $\alpha, \beta = 1, 2, 3$ label the *three* independent components. The magnetic impurity operators $S_{\alpha\beta}$ are generators of the $SU(3)$ Lie algebra for $\alpha, \beta = 1, 2, 3$, provided $S_{11} + S_{22} + S_{33} = 0$; more generally it will be convenient to drop this condition and regard the nine *independent* operators $S_{\alpha\beta}$ as generators of $U(3)$. In the course of the bosonization transcription of the model, it will turn out that the complex phases $\zeta_{\alpha\beta}$, included for generality in the first instance, are all equal, and in fact parametrized in terms of the J_{\parallel} and J_{\perp} couplings. As we shall see, these points will be of significance in the final QDS version, (7). We now turn to a brief discussion of the technicalities of this reformulation.

2.1. Bosonization

The key element in the famous fermion–boson correspondence [6, 8] is the recognition that, for an infinite number of fermionic modes of species α , the bilinear combinations

$$b_{k\alpha}^\dagger = i\sqrt{\frac{2\pi}{Lk}} \sum_{p=-\infty}^{\infty} :c_{p+k\alpha}^\dagger c_{p\alpha}:, \quad b_{k\alpha} = -i\sqrt{\frac{2\pi}{Lk}} \sum_{p=-\infty}^{\infty} :c_{p-k\alpha}^\dagger c_{p\alpha}:$$

fulfil the Heisenberg commutation relations for an infinite set of bosonic modes, namely $[b_{k\alpha}, b_{k'\beta}^\dagger] = \delta_{kk'}\delta_{\alpha\beta}$, and $[b_{k\alpha}, b_{k'\beta}] = 0 = [b_{k\alpha}^\dagger, b_{k'\beta}^\dagger]$, for $k = 2\pi n_k/L$, $n_k = 1, 2, 3, \dots$. Building the appropriate multicomponent local quantum fields in one dimension:

$$\begin{aligned} \psi_\alpha(x) &= \sqrt{\frac{2\pi}{L}} \sum_p e^{-ipx} c_{p\alpha}, & \psi_\alpha^\dagger(x) &= \sqrt{\frac{2\pi}{L}} \sum_p e^{ipx} c_{p\alpha}^\dagger, \\ \varphi_\alpha(x) &= -\sum_{k>0} \sqrt{\frac{2\pi}{Lk}} (e^{-ikx} b_{k\alpha} + e^{ikx} b_{k\alpha}^\dagger) \end{aligned} \tag{4}$$

yields the identification at the level of operators on Fock space

$$\psi_\alpha(x) = \sqrt{\frac{2\pi}{L}} \mathcal{F}_\alpha : e^{-i\varphi_\alpha(x)}:, \tag{5}$$

where the prefactors \mathcal{F}_α , the so-called Klein operators, must fulfil certain additional relations to retain the anti-commutation relations necessary for fermionic operators (see below).

It is evident from (1) and (3) above that the necessity to work with an infinite number of fermionic modes implies that the single particle dispersion relation is extrapolated indefinitely above and below the fermi level. As a consequence, the energy spectrum of the model as a whole is formally unbounded below. In practice this situation is dealt with by introducing a momentum cutoff, which is adequate for most situations in condensed matter. However, in the present context it is crucial to maintain the rigorous mathematical fermion–boson correspondence and isomorphism of Hilbert spaces throughout, and so the regularization of ‘constructive bosonization’ is adopted [8]. This introduces a regularization parameter $a \rightarrow 0$ which sets a scale for the suppression of contributions from wavenumbers $|p| \gtrsim a^{-1}$ away from the fermi surface, by modifying (4) above to

$$\varphi_\alpha(x) = -\sum_{k>0} \sqrt{\frac{2\pi}{Lk}} (e^{-ikx} b_{k\alpha} + e^{ikx} b_{k\alpha}^\dagger) e^{-ak/2}.$$

An important consequence is that normal ordering in operator products can be re-expressed in terms of ordinary products in an expansion in powers of a . In particular, (5) becomes

$$\psi_\alpha(x) = \lim_{a \rightarrow 0} \left(\frac{\mathcal{F}_\alpha}{\sqrt{a}} e^{-i\varphi_\alpha(x)} \right). \tag{6}$$

With these definitions in hand, we can proceed to develop the bosonic counterparts of the various terms in (3) in order to expose the structure of the three-level QDS equivalent.

It should be noted that (5) and (6) are only valid as operator identities when acting on the zero fermion number sectors of the respective fermionic Hilbert spaces. The corrected expressions should have additional charge-dependent phase factors $\exp(2\pi ix/L)^{N_\alpha}$ for charge N_α in each case, which in turn can be seen as deriving from the equivalent formula entailing the number operator \widehat{N}_α ,

$$\widehat{N}_\alpha = \sum_p :c_{p\alpha}^\dagger c_{p\alpha}:$$

applied to each charge eigenspace. Most of the steps in the bosonization transcription entail expressions which are bilinear in fermions, and for field quantities evaluated locally at the magnetic impurity ($x = 0$), so these phases tend to cancel. However, \widehat{N}_α -dependent terms do occur, and their treatment will be taken up in the discussion of the final QDS model below, and technical remarks relegated to appendix A.2.

The systematics by which the couplings and modes are reorganized can be seen by inspecting the ‘magnetic’ term:

$$\sum_{\alpha=1}^3 h_\alpha S_{\alpha\alpha} \equiv h_3 S_3 + h_8 S_8 + h_0 S_0, \quad (H_{(i)}^B)$$

entailing a relabeling from diagonally or doubly indexed quantities $x_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$ to the new set x_A , $A = 3, 8, 0$ (reserving $A = 1, 2, 4, 5, 6, 7$ for off-diagonal labels), and using standard Jacobi three-body combinations:

$$x_3 = \frac{1}{\sqrt{2}}(x_{11} - x_{22}), \quad x_8 = \frac{1}{\sqrt{6}}(x_{11} + x_{22} - 2x_{33}), \quad x_0 = \frac{1}{\sqrt{3}}(x_{11} + x_{22} + x_{33}).$$

The kinetic term is similarly expanded (up to fermion number-dependent terms) as

$$\sum_{p,\alpha=1}^3 \hbar v_F p : c_{p\alpha}^\dagger c_{p\alpha} : = \sum_{k>0} \hbar v_F k (b_{3k}^\dagger b_{3k} + b_{8k}^\dagger b_{8k} + b_{0k}^\dagger b_{0k}).$$

The transverse ‘spin’ interaction terms remain off-diagonal, and are not affected by normal ordering, leading to⁴

$$\frac{L}{2\pi a} J_\perp \sum_{\alpha<\beta} (e^{i\zeta_{\alpha\beta}} e^{-i(\varphi_\alpha(0) - \varphi_\beta(0))} \mathcal{F}_\beta^\dagger \mathcal{F}_\alpha S_{\alpha\beta} + e^{-i\zeta_{\alpha\beta}} e^{i(\varphi_\alpha(0) - \varphi_\beta(0))} \mathcal{F}_\alpha^\dagger \mathcal{F}_\beta S_{\beta\alpha}).$$

By contrast, the longitudinal ‘spin’ couplings (with diagonal fermion bilinears) simply become combinations of the oscillator modes themselves when the bosonization is implemented, in the form

$$J_\parallel \sum_{p,p',\alpha} : c_{p\alpha}^\dagger c_{p'\alpha} : S_{\alpha\alpha} = J_\parallel \sum_{k>0} \sqrt{\frac{kL}{2\pi}} e^{-ka/2} i (S_3 (b_{3k} - b_{3k}^\dagger) + S_8 (b_{8k} - b_{8k}^\dagger) + S_0 (b_{0k} - b_{0k}^\dagger)) \quad (H_{(ii)}^B)$$

—again together with additional terms proportional to fermion number.

2.2. Unitary transformation

These contributions are aggregated together with an additional transformation, a conjugation $U \cdot U^{-1}$ by the operator

$$U = \exp \left(i \sum_{\alpha} \varphi_\alpha(0) S_{\alpha\alpha} \right) \equiv \exp(i(\varphi_3(0) S_3 + \varphi_8(0) S_8 + \varphi_0(0) S_0)).$$

It is evident from the commutation relations of the $U(3)$ Lie algebra, $[S_{\alpha\alpha}, S_{\alpha\beta}] = S_{\alpha\beta}$, $[S_{\beta\beta}, S_{\alpha\beta}] = -S_{\alpha\beta}$ (with $\alpha \neq \beta$) that this unitary transformation will *cancel* the offending scalar exponentials in the transverse coupling terms, leaving the composite operators

$$\frac{L}{2\pi a} J_\perp \sum_{\alpha<\beta} (e^{i\zeta_{\alpha\beta}} \mathcal{F}_\beta^\dagger \mathcal{F}_\alpha S_{\alpha\beta} + e^{-i\zeta_{\alpha\beta}} \mathcal{F}_\alpha^\dagger \mathcal{F}_\beta S_{\beta\alpha}); \quad (H_{(iii)}^B)$$

⁴ Note that a common transcription of the standard two-level Kondo/spin-boson equivalence uses Wannier operator notation, where the coupling constant dimensions are scaled by a factor proportional to the system size (see for example [6]).

the kinetic terms acquire an additional commutator contribution of the same structure as the longitudinal coupling terms, which themselves commute with U :

$$\begin{aligned}
 U \sum_{k>0} \hbar v_F k (b_{3k}^\dagger b_{3k} + b_{8k}^\dagger b_{8k} + b_{0k}^\dagger b_{0k}) U^{-1} &= \sum_{k>0} \hbar v_F k (b_{3k}^\dagger b_{3k} + b_{8k}^\dagger b_{8k} + b_{0k}^\dagger b_{0k}) \\
 &- \hbar v_F \sum_{k>0} \sqrt{\frac{2\pi k}{L}} e^{-ka/2i} (S_3(b_{3k} - b_{3k}^\dagger) + S_8(b_{8k} - b_{8k}^\dagger) + S_0(b_{0k} - b_{0k}^\dagger)).
 \end{aligned}
 \tag{H_{(iv)}^B}$$

(a further term arising from the double commutator in the conjugation by the exponential yields a power series in a whose sum can be removed as an additional overall constant). The outcome of the transcription of the AC–S Hamiltonian (3) is thus the combination of the above ‘magnetic’, longitudinal, transverse and kinetic terms:

$$U \cdot H_{ACS} \cdot U^{-1} = H_{(i)}^B + H_{(ii)}^B + H_{(iii)}^B + H_{(iv)}^B.$$

The reinterpretation of the right-hand side as a dissipative system Hamiltonian proceeds by consideration of the composite operators in $(H_{(iii)}^B)$ which involve the off-diagonal generators $S_{\alpha\beta}$, $\alpha \neq \beta$, of $U(3)$ in combination with Klein operators. It is easily checked using the algebraic properties [8]

$$\mathcal{F}_\alpha \mathcal{F}_\alpha^\dagger = \mathcal{F}_\alpha^\dagger \mathcal{F}_\alpha = 1, \quad \{\mathcal{F}_\alpha^\dagger, \mathcal{F}_\beta\} = \{\mathcal{F}_\alpha, \mathcal{F}_\beta\} = \{\mathcal{F}_\alpha^\dagger, \mathcal{F}_\beta^\dagger\} = 0, \quad \alpha \neq \beta,$$

that, provided that the $S_{\alpha\beta}$ are indeed *elementary* 3×3 matrices, the composite operators defined by $S'_{\alpha\beta} := -\mathcal{F}_\beta^\dagger \mathcal{F}_\alpha S_{\alpha\beta}$, $S'_{\alpha\alpha} := \mathcal{F}_\alpha^\dagger \mathcal{F}_\alpha S_{\alpha\alpha} = S_{\alpha\alpha}$ fulfil the usual $U(3)$ commutation relations, and can be identified with operators acting between the states of the quantum three-level system in the QDS interpretation.

The next step is to combine $H_{(ii)}^B$ and $H_{(iv)}^B$, with the recognition that S_0 is the linear Casimir invariant of $U(3)$ (and is certainly proportional to the 3×3 identity matrix if the original operators are represented with elementary matrices). Thus the terms involving b_{0k} and b_{0k}^\dagger are entirely quadratic and linear—completing the square for such ‘displaced oscillator’ modes enables their contributions to be combined, up to an (infinite) shift in the energy, into a sum of kinetic energy terms for an infinite set of oscillator modes which do not interact with the remainder of the system and can be dropped from the final model. By the same token, the $h_0 S_0$ term can be dropped from $H_{(i)}^B$.

In order to emphasize the similarity between the two-level QDS, the spin- $\frac{1}{2}$ spin-boson model (2), and the new system, we adopt the standard 3×3 Gell–Mann matrices λ_A , $A = 1, \dots, 8$, as an orthogonal basis for the $SU(3)$ generators in the fundamental representation, to play the role of the Pauli matrices in (1). Gathering all terms, the form of the three-level QDS Hamiltonian $U \cdot H_{ACS} \cdot U^{-1} \rightarrow H_{\text{QDS}}^B$ finally becomes

$$\begin{aligned}
 H_{\text{QDS}}^B &:= \varepsilon_3 \lambda_3 + \varepsilon_8 \lambda_8 + \Delta (\lambda_1 + \lambda_4 + \cos \zeta \lambda_6 + \sin \zeta \lambda_7) + \sum_k \hbar \omega_k (b_{k3}^\dagger b_{k3} + b_{k8}^\dagger b_{k8}) \\
 &+ \sum_k \hbar C_{3k} \lambda_3 (b_{k3} + b_{k3}^\dagger) + \hbar C_{8k} \lambda_8 (b_{k8} + b_{k8}^\dagger).
 \end{aligned}
 \tag{7}$$

The QDS parameters have the following definitions in terms of those of the original AC–S model. From above, the detuning parameters ε_3 and ε_8 are simply h_3 and h_8 respectively, and from the kinetic terms the oscillator baths have the frequency spectrum $\omega_k = v_F k$ provided $\omega \ll \omega_c$, where ω_c is the cutoff frequency $\omega_c = v_F/a$. The tunnelling matrix elements are given in terms of the transverse coupling strength of the original model, $\Delta \equiv -J_\perp L/2\pi a$, modulated by a complex phase. By an appropriate basis choice, this phase may be shifted on

to the 2, 3 sector, and expressed in the orthogonal basis by a combination of the corresponding Gell–Mann matrices, namely λ_6 and λ_7 , rotated by angle $\zeta := \zeta_{23} - \zeta_{13} + \zeta_{12}$ (see (3) and also (10), (12) below).

The dissipative terms have been rewritten in the conventional coordinate-coupled form $\cong(b + b^\dagger)$ rather than the imaginary (momentum) combinations $i(b - b^\dagger)$ appearing in the above derivation by means of a canonical transformation $b^\dagger \rightarrow -ib^\dagger, b \rightarrow ib$. Evidently, the overall dissipative couplings are a combination of contributions from different terms, although both coefficients C_{3k} and C_{8k} are equal:

$$C_{3k} = C_{8k} \equiv C_k = -v_F \sqrt{\frac{2\pi k}{L}} e^{-\omega_k/2\omega_c} \left(1 - \frac{J_{\parallel} L}{2\pi \hbar v_F}\right). \quad (8)$$

The spectral frequency $J(\omega)$ follows directly from the definition (in the limit $a \rightarrow 0$). As shown explicitly in appendix A.3, this has the ohmic form

$$J(\omega) = \alpha \omega e^{-\omega/\omega_c}, \quad \text{where} \quad \alpha := \left(1 - \frac{J_{\parallel} L}{2\pi \hbar v_F}\right)^2. \quad (9)$$

The additional parameters emerging from the details of the way the AC–S model and its bosonization are implemented are thus the cutoff frequency ω_c and the dimensionless ohmic coupling α (not to be confused with the spin-label α).

As mentioned above, the fermion-number dependence of the bosonization transcription still requires explanation. Indeed, the introduction of the Klein factor-dependent operators $S'_{\alpha\beta}$ as effective $U(3)$ generators implies that the three states of the quantum system in fact lie across different charge sectors. This situation, and at the same time the treatment of the residual fermion-number-dependent terms, is resolved by noting that the original model (3) has three conserved quantum numbers $\widehat{N}_\alpha + S_{\alpha\alpha}$, $\alpha = 1, 2, 3$, or $\widehat{N}_3 + S_3$, $\widehat{N}_8 + S_8$, and $\widehat{N}_0 + S_0$ in terms of relative degrees of freedom. Of course S_0 is proportional to the identity matrix, and so a projection onto an eigenspace with fixed eigenvalue M_0 is tantamount to fixing the total fermion number at say N_0 which is certainly a conserved quantity. The system further admits a projection onto fixed eigenspaces of the remaining two operators with eigenvalues M_3 and M_8 , say. As shown in appendix A.2, these projections leave the form of (7) unchanged. However, the detuning parameters $\varepsilon_3, \varepsilon_8$ need to be shifted from their original values h_3, h_8 to absorb additional M_3 - and M_8 -dependent contributions.

2.3. Exact solvability and extensions of the model

The equivalence of models (3) and (7) establishes that the three component fermi gas model does indeed have a dissipative system counterpart. The utility of this observation of course derives from also showing that the starting model belongs to the exactly solvable class. The standard coordinate analysis, or an equivalent algebraic formulation in the context of the associated spin chain, requires that the model admit an R -matrix with the appropriate properties. In the present case, following [12], we require that the single particle-impurity scattering matrix S , expressible as the exponential of the interaction component of the Hamiltonian $H_{\text{int}}(J_{\parallel}, J_{\perp})$, can be reparametrized in terms of the R -matrix $R(x^{\alpha=1}, q)$ for some arbitrary but fixed value of the (additive) spectral parameter, say $\alpha = 1$. Thus we demand

$$S = e^{iH_{\text{int}}(J_{\parallel}, J_{\perp})} \equiv R(x^{\alpha=1}, q)$$

in such a way that the parameters x, q become functions of the couplings J_{\parallel}, J_{\perp} . We proceed by an explicit evaluation of S . From (3) we have, using elementary 3×3 matrices $e_{\alpha\beta}$,

$$H_{\text{int}} = J_{\parallel} \sum_{\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + J_{\perp} \sum_{\alpha < \beta} (e^{i\zeta_{\alpha\beta}} e_{\alpha\beta} \otimes e_{\beta\alpha} + e^{-i\zeta_{\alpha\beta}} e_{\beta\alpha} \otimes e_{\alpha\beta}),$$

so

$$\begin{aligned} S = e^{iJ_{\parallel}} \sum_{\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + \cos J_{\perp} \sum_{\alpha \neq \beta} e_{\alpha\alpha} \otimes e_{\beta\beta} \\ + i \sin J_{\perp} \sum_{\alpha < \beta} (e^{i\zeta_{\alpha\beta}} e_{\alpha\beta} \otimes e_{\beta\alpha} + e^{-i\zeta_{\alpha\beta}} e_{\beta\alpha} \otimes e_{\alpha\beta}). \end{aligned} \quad (10)$$

This must be compared with the known forms [13, 14] (see also [15]) for standard trigonometric R -matrices of the appropriate dimension:

$$\begin{aligned} R(x, q) = (qx - q^{-1}x^{-1}) \sum_{\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + (x - x^{-1}) \sum_{\alpha \neq \beta} e_{\alpha\alpha} \otimes e_{\beta\beta} \\ + (q - q^{-1}) \sum_{\alpha < \beta} (xe_{\alpha\beta} \otimes e_{\beta\alpha} + x^{-1}e_{\beta\alpha} \otimes e_{\alpha\beta}). \end{aligned} \quad (11)$$

This expression clearly has the correct structure to be identified with the scattering matrix S if $\zeta_{12} = \zeta_{13} = \zeta_{23}$ with phase factors identified with x . Adopting logarithmic parameters $x = e^{i\bar{f}}$, $q = e^{\bar{\mu}}$, thus with $\zeta \equiv \bar{f}$, the R -matrix is up to a factor of 2:

$$\begin{aligned} R(x, q) = \sinh(i\bar{f} + \bar{\mu}) \sum_{\alpha} e_{\alpha\alpha} \otimes e_{\alpha\alpha} + i \sin(\bar{f}) \sum_{\alpha \neq \beta} e_{\alpha\alpha} \otimes e_{\beta\beta} \\ + \sinh(\bar{\mu}) \sum_{\alpha < \beta} (e^{i\bar{f}} e_{\alpha\beta} \otimes e_{\beta\alpha} + e^{-i\bar{f}} e_{\beta\alpha} \otimes e_{\alpha\beta}). \end{aligned} \quad (12)$$

Comparing the ratios of coefficients in expressions (12) and (10) leads directly to the reparametrization of \bar{f} , $\bar{\mu}$ in terms of J_{\parallel} , J_{\perp} :

$$\cosh \bar{\mu} = \frac{\cos J_{\parallel}}{\cos J_{\perp}}, \quad \cot^2 \bar{f} = \frac{\sin^2 J_{\parallel}}{\sin(J_{\perp} + J_{\parallel}) \sin(J_{\perp} - J_{\parallel})}. \quad (13)$$

With the three-level model admitting a reparametrization showing equivalence to the exactly solvable trigonometric R -matrix, it is clear that the proposed model belongs to this rare and important class of exactly solvable dissipative systems.

3. Discussion

In conclusion, this report has followed the constructive bosonization approach to propose a new exactly solvable three-level quantum dissipative system. Although the background formalism is well known, we have presented concrete details and careful explanations in order to expose the technicalities of the required manipulations, and we anticipate that the methods of this paper may be deployed to find other exactly solvable quantum dissipative system models. In the present case it can be expected that further study will yield insights into the physics of this system as an instance of a QDS model, to be compared and contrasted with the already well-studied mapping of the AKM to the spin- $\frac{1}{2}$ spin-boson model.

As a generalization of the two-level spin-boson/Kondo model correspondence, the three-level analogue presented in this work belongs to the same family of related problems and models. In particular it is interesting to note the relationship between the present model and the triangular lattices and quantum Brownian motion discussed in [16]. It appears that the transverse field terms in (7) are equivalent to hops on this triangular lattice. The model in [16] is shown to correspond to the two-dimensional 3-state Potts model with a boundary, with critical behaviour derivable through $c = 2$ boundary conformal field theory, and it would be instructive to formalize the correspondence to the present three-level dissipative system.

Furthermore, it has been pointed out that the present model bears connections to quantum wire junctions, the dissipative Hofstadter model and open string theory as presented in [17]. The current model contains further generalizations to these systems by including marginal operators coupled to the diagonal elements of the $SU(3)$ algebra.

Further examples of three-level system-environment models to which our new exactly solvable three-level QDS might be applied include three-level quantum dots, single qubit systems addressed by an extra ancillary state, or qutrit states, triatomic triple well potentials, such as ammonia (NH_3) and methyl ($-\text{CH}_3$), as well as Bose–Einstein condensate atomic transistors [18], which have a three well structure. To further develop the model one should resolve the full spectrum and eigenstates of the Hamiltonian via the Bethe *Ansatz* [19], allowing for calculation of dynamical and thermodynamical quantities of interest. One may also be interested in investigating the vacuum sector dependence [8] and finite size effects [20] in the bosonization. The study of entanglement between quantum systems and dissipative environments [21, 22] may also be examined within this impurity-bath system. Generically, it is clear that our analysis of the details of the constructive bosonization and unitary mapping technique suggests that, in the light of potential new applications to QDS, the well-known connections between fermion gas-impurity models and spin chains warrant re-examination.

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Appendix

A.1. Notation

Gell–Mann matrices and the standard form of the Lie algebras of $SU(3)$ and $U(3)$ are based on the (multiplicative) algebra of elementary 3×3 matrices, namely $e_{\alpha\beta}e_{\gamma\delta} = \delta_{\beta\gamma}e_{\alpha\delta}$. Thus the commutation relations are

$$[S_{\alpha\beta}, S_{\gamma\delta}] = \delta_{\beta\gamma}S_{\alpha\delta} - \delta_{\alpha\delta}S_{\gamma\beta}.$$

Introducing the orthogonal basis of trace-normalized λ -matrices via $\frac{1}{2}\text{Tr}(\lambda_A\lambda_B) = \delta_{AB}$, $A, B = 1, 2, \dots, 8$, any 3×3 traceless matrix x can then be expressed in terms of orthogonal coordinates x_A via

$$x_{\alpha\beta} = \frac{1}{2} \sum_{A=1}^8 x_A (\lambda_A)_{\alpha\beta}, \quad x_A = \frac{1}{2} \text{Tr}(x\lambda_A),$$

including of course the elementary matrices themselves. Quantities may also be manipulated using the completeness relation

$$\delta_{\alpha\beta}\delta_{\gamma\delta} = \frac{1}{3}\delta_{\gamma\beta}\delta_{\alpha\delta} + \frac{1}{2} \sum_{A=1}^8 (\lambda_A)_{\gamma\beta}(\lambda_A)_{\alpha\delta},$$

where the right-hand side may be written uniformly over an extended set of λ -matrices λ_A , $A = 0, 1, 2, \dots, 8$, by introducing $\lambda_0 = \sqrt{\frac{2}{3}}1_{3 \times 3}$ to stand in for the identity matrix.

Finally in interpreting one-particle operators in second-quantized form, consider the three states $|\alpha\rangle := c_\alpha^\dagger|0\rangle$ associated with a fixed creation mode (where $\{c_\alpha^\dagger, c_\beta\} = \delta_{\alpha\beta}$ as usual).

Then it is easy to check that

$$\langle \gamma | c_\alpha^\dagger c_\beta | \delta \rangle = \delta_{\delta\alpha} \delta_{\beta\gamma} \equiv (e_{\alpha\beta})_{\delta\gamma}, \quad (\text{A.1})$$

—that is, that the $c_\alpha^\dagger c_\beta$ play the role of elementary matrices on such labelled states. Thus for a term in the particle-impurity interaction such as $\lambda_A \otimes \lambda_A$ we have from above and dropping the \otimes ,

$$\lambda_A \otimes \lambda_A \rightarrow \sum_{\alpha,\beta} \frac{1}{2} \text{Tr}(\lambda_A e_{\alpha\beta}) c_\alpha^\dagger c_\beta \cdot \lambda_A \equiv \frac{1}{2} \sum_{\alpha\beta} c_\alpha^\dagger (\lambda_A)_{\alpha\beta} c_\beta \cdot \lambda_A. \quad (\text{A.2})$$

A.2. Charge sector projection

It was pointed out in the text that the bosonization transcription was carried out to the neglect of various terms accumulating fermion-number (charge) dependent factors. For example the longitudinal couplings certainly amount to a sum over not only the bosonic modes, which is of course one source of the dissipative coupling, but also contain an explicit number operator term. Similarly the standard expression for the bilinear fermion kinetic energy term (involving as it does a derivative of the fermion field, albeit evaluated at zero) is known to contain a term quadratic in the respective charge operators (in fact the coefficients can also differ for different fermionic boundary conditions, but we do not need this option for our basic derivation). Overall we assume that the residual fermion number terms amount to an additional contribution from these sources of

$$C \sum_\alpha \widehat{N}_\alpha^2 + \sum_\alpha C_\alpha \widehat{N}_\alpha \equiv C(\widehat{N}_3^2 + \widehat{N}_8^2) + (C_3 \widehat{N}_3 + C_8 \widehat{N}_8) + (C \widehat{N}_0^2 + C_0 \widehat{N}_0).$$

As mentioned already, total fermion charge is conserved, so for \widehat{N}_0 taken fixed at eigenvalue N_0 say, the last term is an additive constant. For the remaining terms we turn to the relative conserved quantities $\widehat{N}_3 + S_3$, $\widehat{N}_8 + S_8$ and to the projections onto fixed eigenspaces with eigenvalues M_3 , M_8 , respectively. Introduce weight labels $|m, y\rangle$ for the basis of the three-dimensional representation of $SU(3)$ corresponding to the impurity system states, where m , y are the eigenvalues of λ_3 , λ_8 (so that $\frac{1}{2}m$, $\frac{1}{2}y$ are the correctly normalized eigenvalues of $\frac{1}{2}\lambda_3 = S_3$ and $\frac{1}{2}\lambda_8 = S_8$, or isospin and hypercharge, respectively). Imposing the projections, we see that for the total states $|m, y; \psi\rangle$, the fermionic part $|\psi\rangle$ must have charges $N_3 = M_3 - \frac{1}{2}m$, $N_8 = M_8 - \frac{1}{2}y$ and the charge dependent piece on these states becomes

$$C(M_3^2 + M_8^2) + \frac{1}{4}(m^2 + y^2) + (C_3 M_3 + C_8 M_8) - (CM_3 + \frac{1}{2}C_3)m - (CM_8 + \frac{1}{2}C_8)y. \quad (\text{A.3})$$

Finally note that the weight basis of the three-dimensional fundamental representation is $|\pm 1, 1/\sqrt{3}\rangle$ and $|0, -2/\sqrt{3}\rangle$, so that by construction $(m^2 + y^2) \equiv \frac{4}{3}$ for all states. Thus the first line of the transcription is a further additive constant, while the second line amounts to an external ‘magnetic’ coupling and hence an adjustment to the detuning parameters ε_3 , ε_8 , by a shift of $-(CM_3 + \frac{1}{2}C_3)$, $-(CM_8 + \frac{1}{2}C_8)$, respectively.

A.3. Derivation of ohmic coupling

Given the spectrum of bath frequencies $v_F k = \omega_k \equiv \omega_{n_k} = 2\pi v_F/L \cdot n_k$, from the definition of the spectral frequency $J(\omega)$ we have for any test function $f(\omega)$, from (8),

$$\begin{aligned} \int J(\omega) f(\omega) d\omega &= \sum_{n_k} C_k^2 f(\omega_k) = 2\pi v_F/L \sum_{n_k} \alpha \omega_{n_k} e^{-\omega_{n_k}/\omega_c} f(\omega_{n_k}) \\ &\rightarrow 2\pi v_F/L \int dn_k \alpha \omega_{n_k} e^{-\omega_{n_k}/\omega_c} f(\omega_{n_k}) \equiv \int d\omega (\alpha \omega e^{-\omega/\omega_c}) f(\omega), \end{aligned} \quad (\text{A.4})$$

where the approximation that $f(\omega)$ is supported in the region $\omega \ll \omega_c$ has been made. The inferred forms of $J(\omega)$ and α are as given in (9) above.

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